

A generalization of intertwining operators for vertex operator algebras

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Abstract

We generalize the notion of an intertwining operator to \mathbb{N} -graded weak modules over a vertex operator algebra and study their properties. We show a formula for the dimensions of these intertwining operators in terms of modules over the Zhu algebras under some conditions on \mathbb{N} -graded weak modules.

1 Introduction

The purpose of this paper is to generalize the notion of an intertwining operator to \mathbb{N} -graded weak modules over a vertex operator algebra. In the representation theory of groups or Lie algebras, the tensor product of two modules is the tensor product vector space whose module structure is defined by means of a natural coproduct operation, and an intertwining operator is defined to be a module homomorphism from

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the tensor product of two modules to a third module. In contrast, in the representation theory of vertex operator algebras, first the notion of an intertwining operator among an arbitrary triple of modules is defined in [8, Definition 5.4.1], and then the tensor product of two modules is defined by using intertwining operators. Note that the existence of the tensor product of two modules over a vertex operator algebra is not guaranteed in general. The dimension of the space of all intertwining operators among a triple of modules is called the *fusion rule*. It is a natural problem to determine fusion rules for a given vertex operator algebra. In [7, Theorem 1.5.3], Frenkel and Zhu give a formula for the fusion rule among an arbitrary triple of irreducible modules in terms of modules over the Zhu algebra as a generalization of a result in [20] for WZW models. Here we have to be careful that [7, Theorem 1.5.3] is correct for rational vertex operator algebras, however, is not correct for non-rational vertex operator algebras in general as pointed out at the end of [13, Section 2]. A modified result is given in [13, Theorem 2.11]. For many vertex operator algebras, fusion rules among triples of irreducible modules have been determined by using [7, Theorem 1.5.3] and [13, Theorem 2.11] (see for example [1],[2],[3], [16], [18], and [19]).

The definition of an intertwining operator in [8, Definition 5.4.1] makes sense even for weak modules, however, is no longer enough. Actually based on logarithmic conformal field theories in physics, in [14] a generalization of the notion of an intertwining operator, called a *logarithmic intertwining operator*, is given among an arbitrary triple of logarithmic modules by allowing logarithmic terms. Here a logarithmic module is an \mathbb{N} -graded weak module such that each homogeneous space is a generalized $L(0)$ -eigenspace. A generalization of the formula in [7, Theorem 1.5.3] and [13, Theorem 2.11] to logarithmic intertwining operators is given in [11, Theorem 6.6].

The aims of this paper are to introduce a generalization of the notion of an intertwining operator, which I call a *\mathbb{Z} -graded intertwining operator* (see Definition 3.2), among an arbitrary triple of general \mathbb{N} -graded weak modules and to study their properties. For logarithmic modules, a \mathbb{Z} -graded intertwining operator is essentially the same as a logarithmic intertwining operator as we will see later in Section 4. To explain the main idea, we recall a few facts about intertwining operators for (ordinary) modules over a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. For three V -modules $W_i = \bigoplus_{j=0}^{\infty} (W_i)_{\lambda_i+j}$ with lowest weight $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$ and an intertwining operator $I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3\{x\}$,

we define an operator $I^o(u, x) = \sum_{i \in \mathbb{C}} u_i^o x^{-i-1} = x^{\lambda_1 + \lambda_2 - \lambda_3} I(\cdot, x)$, which is already appeared in [7, (1.5.3)], [8, Remark 5.4.4], and [13, (2.12)]. Here $W_3\{x\} = \{\sum_{\alpha \in \mathbb{C}} w_\alpha x^\alpha \mid w_\alpha \in W_3 \ (\alpha \in \mathbb{C})\}$. Then, $I^o(u, x)$ is a map from $W_1 \otimes_{\mathbb{C}} W_2$ to $W_3((x)) = \{\sum_{i \in \mathbb{Z}} w_i x^i \mid w_i \in W_3 \ (i \in \mathbb{Z}) \text{ and } w_i = 0, i \ll 0\}$ and $I(\cdot, x)$ can be written as

$$I(u, x)v = \sum_{i \in \mathbb{Z}} x^{L(0)} (x^{-L(0)} u)_i^o x^{-L(0)} v \quad (1.1)$$

for $u \in W_1$ and $v \in W_2$. Here for the coefficient $L(0)$ of x^{-2} in each $Y_{W_i}(\omega, x)$, $i = 1, 2, 3$, we define

$$x^{\pm L(0)} w = x^{\pm \lambda} w \quad (1.2)$$

for $w \in W_i$ with $L(0)w = \lambda w$, $\lambda \in \mathbb{C}$ and extend $x^{\pm L(0)} w$ for an arbitrary $w \in W_i$, $i = 1, 2, 3$ by linearity. We note that $I^o(\cdot, x)$ satisfies all the conditions in the definition of intertwining operator [8, Definition 5.4.1] except the so called $L(-1)$ -derivative property. For (ordinary) V -modules, $I^o(\cdot, x)$ is nothing but a \mathbb{Z} -graded intertwining operator as we will see later in Proposition 3.3. The main idea is that we redefine $x^{\pm L(0)}$ to be formal variables such that

$$x \frac{d}{dx} x^{\pm L(0)} = x^{\pm L(0)} (\pm L(0)). \quad (1.3)$$

Since this definition of $x^{\pm L(0)}$ makes sense for \mathbb{N} -graded weak modules, we can define “intertwining operators” for \mathbb{N} -graded weak V -modules by using $I^o(\cdot, x)$ and (1.1). Moreover, applying xd/dx to both sides of (1.1) and using Borcherds identity, we automatically get the $L(-1)$ -derivative property.

We expect that various results for intertwining operators can be generalized to \mathbb{Z} -graded intertwining operators. As the main result of this paper I will show a formula for the fusion rules as a generalization of [7, Theorem 1.5.3] and [13, Theorem 2.11] in Theorem 5.3. To state the result precisely, we prepare following symbols. For a vertex operator algebra V and a weak V -module W , $A(V)$ is the Zhu algebra defined in [21, Section 2.1] and $A(W)$ is the $A(V)$ -bimodule defined in [7, Theorem 1.5.1]. For a left $A(V)$ -module U , $S(U) = \oplus_{j=0}^{\infty} S(U)(j)$ is the generalized Verma module with $S(U)(0) = U$ given in [5], $U^* = \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, $S(U)' = \oplus_{j=0}^{\infty} \text{Hom}_{\mathbb{C}}(S(U)(j), \mathbb{C})$ and $I_{\mathbb{Z}} \left(\begin{smallmatrix} S(\Omega_{(3)}^*)' \\ W_1 \ S(\Omega_{(2)}) \end{smallmatrix} \right)$ is the space of all \mathbb{Z} -graded intertwining operators of type $\left(\begin{smallmatrix} S(\Omega_{(3)}^*)' \\ W_1 \ S(\Omega_{(2)}) \end{smallmatrix} \right)$. Now we state the main result:

Theorem 5.3 *For an \mathbb{N} -graded weak V -module W_1 and two left $A(V)$ -modules $\Omega_{(2)}$ and $\Omega_{(3)}$, the map*

$$I_{\mathbb{Z}} \left(\begin{matrix} S(\Omega_{(3)}^*)' \\ W_1 \ S(\Omega_{(2)}) \end{matrix} \right) \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} \Omega_{(2)}, \Omega_{(3)})$$

$$\Phi(\cdot, x) \mapsto o^\Phi \quad (5.46)$$

is a linear isomorphism.

Here we define $o^\Phi(u) = \Phi(u; \deg u - 1)$ for homogeneous $u \in W_1$ and extend $o^\Phi(u)$ for an arbitrary $u \in W_1$ by linearity (see (5.45)). To show the main result, we will modify the proofs of [11, Theorem 6.6], [13, Theorem 2.11], and [21, Theorem 2.2.1] so as not to use the $L(-1)$ -derivative property. For some vertex operator algebras and their modules, we can compute the right-hand side of (5.46). For instance, let us consider Verma modules $M_{c,h}$, $c, h \in \mathbb{C}$ over the Virasoro vertex operator algebra M_c where we use the notation in [7, Section 4]. In this case, the same computation as in [13, Section 2] shows that the right-hand side of (5.46) reduces to the following simple form:

$$\text{Hom}_{A(M_c)}(A(M_{c,h}) \otimes_{A(M_c)} \Omega_{(2)}, \Omega_{(3)}) \cong \text{Hom}_{\mathbb{C}}(\Omega_{(2)}, \Omega_{(3)}).$$

The organization of the paper is as follows. In Section 2 we recall some basic properties of the Zhu algebras, bimodules over the Zhu algebras, and \mathbb{N} -graded weak modules. In Section 3 we recall some basic facts about intertwining operators and introduce the notion of a \mathbb{Z} -graded intertwining operator. In Section 4 for an arbitrary triple of logarithmic modules $W_i, i = 1, 2, 3$, we construct a linear isomorphism from the space of all logarithmic intertwining operators of type $\begin{pmatrix} W_3 \\ W_1 \ W_2 \end{pmatrix}$ to the space of all \mathbb{Z} -graded intertwining operators of the same type. In Section 5 we will show the main result. In Section 6 we list some notations.

2 Preliminary

We assume that the reader is familiar with the basic knowledge on vertex algebras as presented in [4], [5], and [12]. Throughout this paper, \mathbb{N} denotes the set of all non-negative integers, $x, y, x_0, x_1, x_2, \dots$ are commutative formal variables, and $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra. For the Virasoro element ω of V and a weak V -module

(M, Y_M) , we write

$$Y_M(\omega, x) = \sum_{i \in \mathbb{Z}} L(i) x^{-i-2}. \quad (2.1)$$

We recall some properties of the Zhu algebra $A(V)$ of V and the $A(V)$ -bimodules associated with weak modules from [21, Section 2] and [7, Section 1], and [13, Section 2]. Let M be a weak V -module. For homogeneous $a \in V$ and $u \in M$, we define

$$a \circ u = \text{Res}_x (1+x)^{\text{wt } a} x^{-2} Y_M(a, x) u \in M \quad (2.2)$$

and

$$a * u = \text{Res}_x (1+x)^{\text{wt } a} x^{-1} Y_M(a, x) u \in M, \quad (2.3)$$

$$u * a = \text{Res}_x (1+x)^{\text{wt } a-1} x^{-1} Y_M(a, x) u \in M. \quad (2.4)$$

Here, Res_x is defined by

$$\text{Res}_x f(x) = f_{-1} \quad (2.5)$$

for $f(x) = \sum_{i \in \mathbb{Z}} f_i x^i \in M[[x, x^{-1}]]$. We extend (2.2)–(2.4) for an arbitrary $a \in V$ by linearity. We also define

$$O(M) = \text{Span}_{\mathbb{C}} \{a \circ u \mid a \in V, u \in M\} \quad (2.6)$$

and take the following quotient space:

$$A(M) = M/O(M). \quad (2.7)$$

If one takes $M = V$, then $A(V)$ is an associative \mathbb{C} -algebra, called the *Zhu algebra* of V , with multiplication (2.3) by [21, Theorem 2.1.1]. It follows from [7, Theorem 1.5.1] that $A(M)$ is an $A(V)$ -bimodule under the actions (2.3) and (2.4). It follows from the proof of [21, Lemma 2.1.2] that for homogeneous $a \in V$ and $u \in M$,

$$\text{Res}_x (1+x)^{\text{wt } a} x^k Y_M(a, x) u \in O(M) \text{ for } k \leq -2. \quad (2.8)$$

We shall use elements of M to represent elements of $A(M)$. For a left $A(V)$ -module U and $a \in A(V)$, we denote the action of a on U by $o(a)$:

$$\begin{aligned} A(V) &\rightarrow \text{End}_{\mathbb{C}}(U) \\ a &\mapsto o(a). \end{aligned} \quad (2.9)$$

A weak V -module M is called \mathbb{N} -graded if M admits a decomposition $M = \bigoplus_{j=0}^{\infty} M(j)$ such that

$$a_k M(j) \subset M(\text{wt } a + j - k - 1) \quad (2.10)$$

for homogeneous $a \in V$, $j \in \mathbb{N}$, and $k \in \mathbb{Z}$. For an \mathbb{N} -graded weak V -module $M = \bigoplus_{j=0}^{\infty} M(j)$ and $u \in M(j)$, $j \in \mathbb{N}$, we define the *degree* of u by

$$\deg u = j. \quad (2.11)$$

Following [5], an \mathbb{N} -graded weak V -module $M = \bigoplus_{i=0}^{\infty} M(i)$ is called a *generalized Verma V -module* if M is generated by $M(0)$ and for every \mathbb{N} -graded weak V -module W and every $A(V)$ -module homomorphism

$$f : M(0) \rightarrow \{w \in W \mid a_i w = 0 \text{ for homogeneous } a \in V \text{ and } i \geq \text{wt } a\}, \quad (2.12)$$

there exists a unique V -module homomorphism $F : M \rightarrow W$ such that $F|_{M(0)} = f$. For an arbitrary $A(V)$ -module U , [5, Theorem 6.2] shows there exists a unique generalized Verma V -module $S(U)$ with $S(U)(0) = U$ up to isomorphism, where $S(U)$ is denoted by $\bar{M}(U)$ in [5].

3 \mathbb{Z} -graded intertwining operators

In this section we first recall the definition of an intertwining operator from [8, Definition 5.4.1] and then introduce the notion of a \mathbb{Z} -graded intertwining operator as a generalization of an intertwining operator. For a vector space M over \mathbb{C} and $p, q \in \mathbb{Z}$, we define

$$\begin{aligned} M[x, x^{-1}]_{[p, q]} &= \left\{ \sum_{i=p}^q u_i x^i \mid u_p, u_{p+1}, \dots, u_q \in M \right\}, \\ M\{x\} &= \left\{ \sum_{\alpha \in \mathbb{C}} u_{\alpha} x^{\alpha} \mid u_{\alpha} \in M \ (\alpha \in \mathbb{C}) \right\}, \\ M((x)) &= \left\{ \sum_{i \in \mathbb{Z}} u_i x^i \mid u_i \in M \ (i \in \mathbb{Z}) \text{ and } u_i = 0, i \ll 0 \right\}, \text{ and} \\ M[[x, y]] &= \left\{ \sum_{i, j=0}^{\infty} u_{ij} x^i y^j \mid u_{ij} \in M \ (i, j \in \mathbb{N}) \right\}. \end{aligned} \quad (3.1)$$

Define three linear injective maps

$$\begin{aligned}\iota_{x,y} : M[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}] &\rightarrow M((x))((y)), \\ \iota_{y,x} : M[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}] &\rightarrow M((y))((x)), \text{ and} \\ \iota_{x,y-x} : M[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}] &\rightarrow M((x))((y-x))\end{aligned}\quad (3.2)$$

determined by

$$\begin{aligned}\iota_{x,y}(x^j y^k (x-y)^l) &= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^i x^{j+l-i} y^{k+i}, \\ \iota_{y,x}(x^j y^k (x-y)^l) &= \sum_{i=0}^{\infty} \binom{l}{i} (-1)^{l-i} y^{k+l-i} x^{j+i}, \text{ and} \\ \iota_{x,y-x}(x^j y^k (x-y)^l) &= \sum_{i=0}^{\infty} \binom{k}{i} x^{j+k-i} (-1)^l (y-x)^{l+i}\end{aligned}\quad (3.3)$$

for $j, k, l \in \mathbb{Z}$ and $\iota_{x,y}(u) = \iota_{y,x}(u) = \iota_{x,y-x}(u) = u$ for $u \in M$.

We recall the definition of an intertwining operator from [8, Definition 5.4.1].

Definition 3.1. Let W_1, W_2 , and W_3 be three V -modules. An *intertwining operator* of type $\begin{pmatrix} W_3 \\ W_1 \quad W_2 \end{pmatrix}$ is a linear map

$$\begin{aligned}I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 &\rightarrow W_3\{x\} \\ I(u, x)v &= \sum_{\alpha \in \mathbb{C}} u_{\alpha} v x^{-\alpha-1}, \\ u &\in W_1, v \in W_2, \text{ and } u_{\alpha} \in \text{Hom}_{\mathbb{C}}(W_2, W_3),\end{aligned}\quad (3.4)$$

such that the following conditions are satisfied:

- (1) For $u \in W_1, v \in W_2$, and $\alpha \in \mathbb{C}$,

$$u_{\alpha+m} v = 0 \text{ for sufficiently large } m \in \mathbb{N}. \quad (3.5)$$

- (2) For $u \in W_1$ and $a \in V$,

$$\begin{aligned}x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(a, x_1) I(u, x_2) &- x_0^{-1} \delta\left(\frac{x_2 - x_1}{x_0}\right) I(u, x_2) Y(a, x_1) \\ &= x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) I(Y(a, x_0) u, x_2).\end{aligned}\quad (3.6)$$

(3) ($L(-1)$ -derivative property) For $u \in W_1$,

$$I(L(-1)u, x) = \frac{d}{dx}I(u, x). \quad (3.7)$$

We denote by $I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ the space of all intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ and call its dimension the *fusion rule* of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$. A formula for the fusion rule among an arbitrary triple of irreducible modules is given in [7, Theorem 1.5.3] and [13, Theorem 2.11].

For a vector space U , we define a subspace $U[\{x\}]$ of $U\{x\}$ by

$$U[\{x\}] = \left\{ \sum_{\alpha \in \mathbb{C}} u_{\alpha} x^{\alpha} \mid \begin{array}{l} u_{\alpha} \in U \ (\alpha \in \mathbb{C}) \text{ and for any } \alpha \in \mathbb{C}, \\ w_{\alpha+i} = 0 \text{ for sufficiently small } i \in \mathbb{Z} \end{array} \right\}. \quad (3.8)$$

Standard arguments (cf. [12, Sections 3.2–3.4] and [17, Lemma 2.4]) show that the condition (2) in Definition 3.1 is equivalent to the following condition: For $u \in W_1, v \in W_2$ and $a \in V$, there exists

$$I(a, u, v|x, y) \in W_3[[y]][\{x\}][y^{-1}, (x-y)^{-1}] \quad (3.9)$$

such that

$$\begin{aligned} \iota_{x,y}I(a, u, v|x, y) &= I(u, x)Y_{W_2}(a, y)v, \\ \iota_{y,x}I(a, u, v|x, y) &= Y_{W_3}(a, y)I(u, x)v, \quad \text{and} \\ \iota_{x,y-x}I(a, u, v|x, y) &= I(Y_{W_1}(a, y-x)u, x)v. \end{aligned} \quad (3.10)$$

Let $I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3\{x\}$ be an intertwining operator. For $\alpha \in \mathbb{C}$, by taking a to be the Virasoro element $\omega \in V$ in (3.6) and comparing the coefficients of $x_0^{-1}x_1^{-2}x_2^{-\alpha-1}$ in both sides, we have

$$(L(-1)u)_{\alpha+1} = L(0)u_{\alpha} - (L(0)u)_{\alpha} - u_{\alpha}L(0) \quad (3.11)$$

and therefore the $L(-1)$ -derivative property (3.7) can be replaced by the following condition:

$$x \frac{d}{dx}I(u, x) = L(0)I(u, x) - I(L(0)u, x) - I(u, x)L(0). \quad (3.12)$$

Suppose $W_i, i = 1, 2, 3$ admit decompositions

$$W_i = \bigoplus_{j=0}^{\infty} (W_i)_{\lambda_i+j} \quad (3.13)$$

where $(W_i)_{\lambda_i+j}$ is the eigenspace for $L(0)$ with eigenvalue λ_i+j , $j \in \mathbb{N}$. It follows by (3.12) that

$$u_\alpha v = 0 \text{ for } \alpha \notin \lambda_1 + \lambda_2 - \lambda_3 + \mathbb{Z} \quad (3.14)$$

and

$$u_{\lambda_1+\lambda_2-\lambda_3+k}(W_2)_{\lambda_2+j} \subset (W_3)_{\lambda_3+i+j-k-1} \quad (3.15)$$

for $u \in (W_1)_{\lambda_1+i}$, $j \in \mathbb{N}$, and $k \in \mathbb{Z}$. The properties (3.14) and (3.15) are essentially used in the proof of the formula for the fusion rules given in [7, Theorem 1.5.3] and [13, Theorem 2.11].

The definition of an intertwining operator in [8, Definition 5.4.1] makes sense even for weak V -modules, however, the condition (3.12) seems to be too strong for weak V -modules as explained below. Let $W_i, i = 1, 2, 3$ be three weak V -modules and $I(\cdot, x) : W_1 \otimes W_2 \rightarrow W_3\{x\}$ a linear map which satisfies all the conditions in Definition 3.1. Despite the action of $L(0)$ on a weak V -module is not necessarily semisimple, for $u \in W_1$ and $v \in W_2$, (3.12) forces that each coefficient of

$$L(0)I(u, x)v - I(L(0)u, x)v - I(u, x)L(0)v \quad (3.16)$$

is a scalar multiple of the corresponding coefficient of $I(u, x)v$. Moreover, we can not expect a generalization of the formula for the fusion rules given in [7, Theorem 1.5.3] and [13, Theorem 2.11] because similar conditions like (3.14) and (3.15), which are essential for these results, do not follow from (3.12). Thus, we need to modify the condition (3.12).

To do that, we return to the case of intertwining operators $I(\cdot, x)$ among a triple of V -modules W_1, W_2 and W_3 as in (3.13). We define a map

$$I^o(\cdot, x) = x^{\lambda_1+\lambda_2-\lambda_3} I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3((x))$$

$$u \otimes v \mapsto \sum_{i \in \mathbb{Z}} u_i^o x^{-i-1}, \quad (3.17)$$

which is already appeared in [7, (1.5.3)], [8, Remark 5.4.4], and [13, (2.12)], and we denote $(W_i)_{\lambda_i+j}$ by $W_i(j)$ for $i = 1, 2, 3$ and $j \in \mathbb{N}$. Then, we have

$$\begin{aligned} I(u, x)v &= x^{-\lambda_1-\lambda_2+\lambda_3} I^o(u, x)v \\ &= \sum_{i \in \mathbb{Z}} x^{L(0)} (x^{-L(0)} u)_i^o x^{-L(0)} v \end{aligned} \quad (3.18)$$

for $u \in W_1$ and $v \in W_2$. Here we define

$$x^{\pm L(0)}w = x^{\pm \lambda}w \quad (3.19)$$

for $w \in W_i, i = 1, 2, 3$ with $L(0)w = \lambda w, \lambda \in \mathbb{C}$ and extend $x^{\pm L(0)}w$ for an arbitrary $w \in W_i, i = 1, 2, 3$ by linearity. The map $I^o(\cdot, x)$ satisfies (3.5), (3.6), and

$$u_k^o W_2(j) \subset W_3(i + j - k - 1) \quad (3.20)$$

for $u \in W_1(i), j \in \mathbb{N}$, and $k \in \mathbb{Z}$ by (3.15). Based on the properties (3.5), (3.6), and (3.20) of $I^o(\cdot, x)$, we introduce the following notion:

Definition 3.2. Let W_1, W_2 and W_3 be three \mathbb{N} -graded weak V -modules and $\Phi(\cdot, x) = \sum_{n \in \mathbb{Z}} \Phi(\cdot; n)x^{-n-1}$ a linear map from $W_1 \otimes_{\mathbb{C}} W_2$ to $W_3((x))$. We call Φ a \mathbb{Z} -graded intertwining operator of type $\binom{W_3}{W_1 W_2}$ if

- (1) For $i, j \in \mathbb{N}, k \in \mathbb{Z}$, and $u \in W_1(i)$,

$$\Phi(u; k)W_2(j) \subset W_3(i + j - k - 1). \quad (3.21)$$

- (2) For $u \in W_1, v \in W_2$, and $a \in V$, there exists

$$\Phi(a, u, v|x, y) \in W_3[[x, y]][x^{-1}, y^{-1}, (x - y)^{-1}] \quad (3.22)$$

such that

$$\begin{aligned} \iota_{x,y} \Phi(a, u, v|x, y) &= \Phi(u, x)Y_{W_2}(a, y)v, \\ \iota_{y,x} \Phi(a, u, v|x, y) &= Y_{W_3}(a, y)\Phi(u, x)v, \quad \text{and} \\ \iota_{x,y-x} \Phi(a, u, v|x, y) &= \Phi(Y_{W_1}(a, y - x)u, x)v. \end{aligned} \quad (3.23)$$

Standard arguments (cf. [12, Sections 3.2–3.4] and [17, Lemma 2.4]) show that the condition (2) in Definition 3.2 is equivalent to the following *Borcherds identity*: for $a \in V, u \in W_1, v \in W_2$, and $l, m, n \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{i=0}^{\infty} \binom{m}{i} \Phi(a_{l+i}u; m + n - i)v \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (a_{m+l-i} \Phi(u; n + i)v + (-1)^{l+1} \Phi(u; m + i)a_{n+l-i}v). \end{aligned} \quad (3.24)$$

We denote by $I_{\mathbb{Z}}\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ the space of all \mathbb{Z} -graded intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$. The following result shows that for V -modules a \mathbb{Z} -graded intertwining operator is essentially the same as an intertwining operator.

Proposition 3.3. *For three V -modules W_1 , W_2 , and W_3 , the map*

$$\begin{aligned} I\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right) &\rightarrow I_{\mathbb{Z}}\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right) \\ I(\ , x) &\mapsto I^o(\ , x) \end{aligned} \quad (3.25)$$

is a linear isomorphism.

Proof. We may assume that W_1 , W_2 , and W_3 admit decompositions as in (3.13). We have already shown before Definition 3.2 that $I^o(\ , x)$ is a \mathbb{Z} -graded intertwining operator for an intertwining operator $I(\ , x)$. For a \mathbb{Z} -graded intertwining operator $\Phi(\ , x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3((x))$, $x^{-\lambda_1-\lambda_2+\lambda_3}\Phi(\ , x)$ satisfies (3.5) and (3.6). The same argument as in (3.11) shows

$$\Phi(L(-1)u; i+1) = L(0)\Phi(u; i) - \Phi(L(0)u; i) - \Phi(u; i)L(0) \quad (3.26)$$

for $u \in W_1$ and $i \in \mathbb{Z}$, which implies

$$\frac{d}{dx}(x^{-\lambda_1-\lambda_2+\lambda_3}\Phi(u, x)) = x^{-\lambda_1-\lambda_2+\lambda_3}\Phi(L(-1)u, x) \quad (3.27)$$

by (3.21). Therefore $x^{-\lambda_1-\lambda_2+\lambda_3}\Phi(u, x)$ is an intertwining operator and this completes the proof. \square

As we will see later in Proposition 4.3, the isomorphism (3.25) is generalized to the case of logarithmic intertwining operators introduced in [14].

We note that the $L(-1)$ -derivative property (3.7), or equivalently (3.12), is not required for \mathbb{Z} -graded intertwining operators. However, the following modifications of \mathbb{Z} -graded intertwining operators satisfy (3.12). We redefine $x^{L(0)}$ and $x^{-L(0)}$ to be two formal variables and let $\mathbb{C}x^{L(0)}$ (resp. $\mathbb{C}x^{-L(0)}$) be a vector space with a basis $x^{L(0)}$ (resp. $x^{-L(0)}$). For a $L(0)$ -module W , we define vector spaces

$$\begin{aligned} x^{L(0)}W &= \mathbb{C}x^{L(0)} \otimes_{\mathbb{C}} W, \\ x^{-L(0)}W &= \mathbb{C}x^{-L(0)} \otimes_{\mathbb{C}} W \end{aligned} \quad (3.28)$$

and a linear map

$$\begin{aligned} x \frac{d}{dx} : x^{\pm L(0)} W &\rightarrow x^{\pm L(0)} W \\ x^{\pm L(0)} \otimes u &\mapsto x^{\pm L(0)} \otimes L(0)u, \quad u \in W. \end{aligned} \quad (3.29)$$

If W is a weak V -module, then so are $x^{\pm L(0)} W$ by defining

$$Y_{x^{\pm L(0)} W}(a, y)(x^{\pm L(0)} \otimes u) = x^{\pm L(0)} \otimes Y_W(a, y)u \quad (3.30)$$

for $a \in V$ and $u \in W$. Clearly $x^{\pm L(0)} W$ are isomorphic to W . For three $L(0)$ -modules $W_i, i = 1, 2, 3$ and a linear map $f : x^{-L(0)} W_1 \otimes_{\mathbb{C}} x^{-L(0)} W_2 \rightarrow x^{L(0)} W_3$, we define a map

$$x \frac{d}{dx} f : x^{-L(0)} W_1 \otimes_{\mathbb{C}} x^{-L(0)} W_2 \rightarrow x^{L(0)} W_3 \quad (3.31)$$

by

$$\begin{aligned} &(x \frac{d}{dx} f)(p \otimes q) \\ &= x \frac{d}{dx} (f(p \otimes q)) + f((x \frac{d}{dx} p) \otimes q) + f(p \otimes x \frac{d}{dx} q) \end{aligned} \quad (3.32)$$

for $p \in x^{-L(0)} W_1$ and $q \in x^{-L(0)} W_2$. For three \mathbb{N} -graded weak V -modules W_1, W_2 , and W_3 , a \mathbb{Z} -graded intertwining operator $\Phi : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3((x))$, and $i \in \mathbb{Z}$, we define a map

$$\begin{aligned} \hat{\Phi}_i : x^{-L(0)} W_1 \otimes_{\mathbb{C}} x^{-L(0)} W_2 &\rightarrow x^{L(0)} W_3 \\ (x^{-L(0)} \otimes u) \otimes (x^{-L(0)} \otimes v) &\mapsto x^{L(0)} \otimes \Phi(u; i)v. \end{aligned} \quad (3.33)$$

Then the sequence $(\hat{\Phi}_i)_{i \in \mathbb{Z}}$ satisfies (3.24) and

$$\hat{\Phi}_k((x^{-L(0)} \otimes u) \otimes (x^{-L(0)} \otimes v)) \in x^{L(0)} W_3(i + j - k) \quad (3.34)$$

for $k \in \mathbb{Z}$, $u \in W_1(i)$, and $v \in W_2(j)$, which is an analogue of (3.21). By (3.32), we automatically have the following analogue of the $L(-1)$ -derivative property (3.7) (or (3.12)).

Lemma 3.4. *For $i \in \mathbb{Z}$, $u \in W_1$, and $v \in W_2$, we have*

$$(x \frac{d}{dx} \hat{\Phi}_i)(u \otimes v) = L(0) \hat{\Phi}_i(u \otimes v) - \hat{\Phi}_i(L(0)u \otimes v) - \hat{\Phi}_i(u \otimes L(0)v). \quad (3.35)$$

4 A relation between logarithmic intertwining operators and \mathbb{Z} -graded intertwining operators

In this section we will show that for logarithmic modules, a \mathbb{Z} -graded intertwining operator is essentially the same as a logarithmic intertwining operator introduced in [14]. Throughout this section we assume all weak V -modules M satisfy the following condition: there exists $\lambda \in \mathbb{C}$ such that M admits a decomposition

$$\begin{aligned} M &= \bigoplus_{i=0}^{\infty} M_{\lambda+i}, \\ M_h &= \{u \in M \mid (L(0) - h)^n u = 0 \text{ for some } n \in \mathbb{Z}_{>0}\} \\ &\text{with } \dim_{\mathbb{C}} M_h < \infty \text{ for } h \in \lambda + \mathbb{Z}. \end{aligned} \quad (4.1)$$

Finite direct sums of weak V -modules satisfies the condition above are called *logarithmic* V -modules in [14]. We recall the definition of logarithmic intertwining operators from [14, Definition 1.3] and [10, Definition 3.7].

Definition 4.1. Let $W_i = \bigoplus_{j=0}^{\infty} (W_i)_{\lambda_i+j}$, $i = 1, 2, 3$ be three weak V -modules which satisfy (4.1). A *logarithmic intertwining operator* is a linear map

$$\begin{aligned} I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 &\rightarrow W_3[\log x]\{x\} \\ I(u, x)v &= \sum_{\alpha \in \mathbb{C}} \sum_{n=0}^{\infty} u_{\alpha,n} v x^{-\alpha-1} (\log x)^n, \\ u &\in W_1, v \in W_2, \text{ and } u_{n,\alpha} \in \text{Hom}_{\mathbb{C}}(W_2, W_3) \end{aligned} \quad (4.2)$$

such that the following conditions are satisfied:

- (1) For $u \in W_1$, $v \in W_2$, and $\alpha \in \mathbb{C}$,

$$u_{\alpha+m,k} v = 0 \text{ for sufficiently large } m \in \mathbb{N}. \quad (4.3)$$

- (2) For $u \in W_1$ and $a \in V$,

$$\begin{aligned} &x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y(a, x_1) I(u, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{x_0}\right) I(u, x_2) Y(a, x_1) \\ &= x_1^{-1} \delta\left(\frac{x_2 + x_0}{x_1}\right) I(Y(a, x_0)u, x_2). \end{aligned} \quad (4.4)$$

(3) For $u \in W_1$,

$$I(L(-1)u, x) = \frac{d}{dx}I(u, x). \quad (4.5)$$

We denote by $I_{\log} \left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$ the space of all logarithmic intertwining operators of type $\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix} \right)$. We recall some basic properties about logarithmic intertwining operators from [10],[14], and [15]. The same argument as in (3.14) shows

$$u_{\alpha,n}v = 0 \text{ for } \alpha \notin \lambda_1 + \lambda_2 - \lambda_3 + \mathbb{Z}. \quad (4.6)$$

For a logarithmic intertwining operator $I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3[\log x]\{x\}$, we write

$$I(\cdot, x) = \sum_{i=0}^{\infty} I^{(i)}(\cdot, x)(\log x)^i, \quad I^{(i)}(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3\{x\} \quad (4.7)$$

and define

$$\begin{aligned} I^o(\cdot, x) &= \sum_{i \in \mathbb{Z}} I^o(\cdot; i)x^{-i-1}, \quad I^o(\cdot; i) \in \text{End}_{\mathbb{C}}(W_1 \otimes_{\mathbb{C}} W_2, W_3) \\ &= x^{\lambda_1 + \lambda_2 - \lambda_3} I^{(0)}(\cdot, x) \in (\text{End}_{\mathbb{C}}(W_2, W_3))[[x, x^{-1}]]. \end{aligned} \quad (4.8)$$

Note that

$$I^o(u; i) = u_{i+\lambda_1+\lambda_2-\lambda_3,0} \quad (4.9)$$

for $u \in W_1$ and $i \in \mathbb{Z}$. It follows by (4.5) that

$$xI^{(i)}(L(-1)u, x) = x \frac{d}{dx} I^{(i)}(u, x) + (i+1)I^{(i+1)}(u, x) \quad (4.10)$$

for all $i \in \mathbb{N}$ and therefore $I(\cdot, x)$ is uniquely determined by $I^{(0)}(\cdot, x)$, or $I^o(\cdot, x)$.

For a weak V -module $M = \bigoplus_{i=0}^{\infty} M_{\lambda+i}$ as in (4.1), M is an \mathbb{N} -graded weak V -module with $M(i) = M_{\lambda+i}$ for $i \in \mathbb{N}$ and one can take the Jordan decomposition

$$L(0) = S + N \quad (4.11)$$

of $L(0)$ on M where S is the semisimple part of $L(0)$ and N is the nilpotent part of $L(0)$. For $u \in M$ such that $Su = \lambda u$, $\lambda \in \mathbb{C}$, we define

$$x^S u = x^{\lambda} u. \quad (4.12)$$

We extend x^S for an arbitrary $u \in M$ by linearity. For $u \in M$ we define

$$x^N u = e^{N \log x} u = \sum_{i=0}^{\infty} \frac{(\log x)^i N^i}{i!} u \quad \text{and} \\ x^{L(0)} u = x^S x^N u. \quad (4.13)$$

We also define $x^{-L(0)} u$ by the same manner. For $u \in M$ we clearly have

$$x \frac{d}{dx} x^{\pm L(0)} u = x^{\pm L(0)} (\pm L(0)) u. \quad (4.14)$$

Although the following generalization of (3.18) seems to be well known, we give a proof.

Lemma 4.2. *Let $W_i, i = 1, 2, 3$ be as above. For a logarithmic intertwining operator $I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3[\log x]\{x\}$, we have*

$$I(u, x)v = \sum_{i \in \mathbb{Z}} x^{L(0)} I^o(x^{-L(0)} u; i) x^{-L(0)} v \quad (4.15)$$

for $u \in W_1$ and $v \in W_2$, where the actions of $x^{\pm L(0)}$ on $W_i, i = 1, 2, 3$ are defined by (4.13).

Proof. For $u \in W_1$ and $v \in W_2$, we denote by $J(u, x)v$ the right-hand side of (4.15). By (4.13), $J(u, x)v$ can be written as

$$J(u, x)v = \sum_{i \in \mathbb{Z}} e^{N \log x} I^o(e^{-N \log x} u; i) e^{-N \log x} v x^{\lambda_3 - \lambda_1 - \lambda_2 - i - 1} \\ \in W_3[\log x]\{x\}. \quad (4.16)$$

and therefore $J(u, x)v$ satisfies (4.3).

Since N is a V -module homomorphism by [10, Proposition 2.2], $J(\cdot, x)$ satisfies (4.4). By (4.14), we have (4.5) as follows:

$$x \frac{d}{dx} J(u, x)v \\ = \sum_{i \in \mathbb{Z}} (x^{L(0)} L(0) I^o(x^{-L(0)} u; i) x^{-L(0)} v - x^{L(0)} I^o(x^{-L(0)} L(0) u; i) x^{-L(0)} v \\ - x^{L(0)} I^o(x^{-L(0)} u; i) x^{-L(0)} L(0) v) \\ = \sum_{i \in \mathbb{Z}} x^{L(0)} I^o(x^{-L(0)} L(-1) u; i + 1) x^{-L(0)} v. \quad (4.17)$$

Thus, $J(\cdot, x)$ is a logarithmic intertwining operator. If we write

$$J(\cdot, x) = \sum_{i=0}^{\infty} J^{(i)}(\cdot, x)(\log x)^i, \quad J^{(i)}(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3\{x\}, \quad (4.18)$$

then we have $J^{(0)}(\cdot, x) = I^{(0)}(\cdot, x)$ by (4.16) and therefore $I(\cdot, x) = J(\cdot, x)$ by the comment right after (4.10). \square

For a logarithmic intertwining operator $I(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3[\log x]\{x\}$, $I^o(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3((x))$ is a \mathbb{Z} -graded intertwining operator since $I^o(\cdot, x)$ satisfies (3.6). Conversely, the proof of Lemma 4.2 shows that for a \mathbb{Z} -graded intertwining operator $\Phi(\cdot, x) : W_1 \otimes_{\mathbb{C}} W_2 \rightarrow W_3((x))$, the map

$$\begin{aligned} W_1 \otimes_{\mathbb{C}} W_2 &\rightarrow W_3[\log x]\{x\} \\ u \otimes v &\mapsto \sum_{i \in \mathbb{Z}} x^{L(0)} \Phi(x^{-L(0)} u; i) x^{-L(0)} v \end{aligned} \quad (4.19)$$

is a logarithmic intertwining operator. Thus we have the following result.

Proposition 4.3. *Let W_1, W_2 , and W_3 be three weak V -modules which satisfy (4.1). Then, the map*

$$\begin{aligned} I_{\log} \left(\begin{matrix} W_3 \\ W_1 \ W_2 \end{matrix} \right) &\rightarrow I_{\mathbb{Z}} \left(\begin{matrix} W_3 \\ W_1 \ W_2 \end{matrix} \right) \\ I(\cdot, x) &\mapsto I^o(\cdot, x) \end{aligned} \quad (4.20)$$

is a linear isomorphism.

5 The main theorem

Throughout this section $\Omega_{(2)}$ and $\Omega_{(3)}$ are two left $A(V)$ -modules and W_1 is an \mathbb{N} -graded weak V -module. In this section we establish a one-to-one correspondence between $\text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} \Omega_{(2)}, \Omega_{(3)})$ and $I_{\mathbb{Z}} \left(\begin{matrix} S(\Omega_{(3)}^*)' \\ W_1 \ S(\Omega_{(2)}) \end{matrix} \right)$ as a generalization of [7, Theorem 1.5.3] and [13, Theorem 2.11]. Here for a left $A(V)$ -module U , $S(U) = \bigoplus_{j=0}^{\infty} S(U)(j)$ is the generalized Verma V -module with $S(U)(0) = U$ defined in Section 2, $U^* = \text{Hom}_{\mathbb{C}}(U, \mathbb{C})$, and $S(U)' = \bigoplus_{j=0}^{\infty} \text{Hom}_{\mathbb{C}}(S(U)(j), \mathbb{C})$. We

will show this result by modifying the proofs of [11, Theorem 6.6], [13, Theorem 2.11], and [21, Theorem 2.2.1] so as not to use the $L(-1)$ -derivative property.

For a vector space U , $T(U)$ denotes the tensor algebra of U . For an \mathbb{N} -graded weak V -module W_1 , $T(W_1, \Omega_{(2)})$ denotes the tensor algebra $T((V \oplus W_1 \oplus \Omega_{(2)})[t, t^{-1}])$ and $F(W_1, \Omega_{(2)})$ denotes the subspace $T(V[t, t^{-1}]) \otimes_{\mathbb{C}} W_1[t, t^{-1}] \otimes_{\mathbb{C}} T(V[t, t^{-1}]) \otimes_{\mathbb{C}} \Omega_{(2)}$ of $T(W_1, \Omega_{(2)})$. For simplicity we shall omit the tensor product symbol. For $a \in V \oplus W_1 \oplus \Omega_{(2)}$ and $i \in \mathbb{Z}$, $a(i)$ denotes $a \otimes t^i$. For $a \in V \oplus W_1 \oplus \Omega_{(2)}$, we define a map

$$Y_{T(W_1, \Omega_{(2)})}(a, x) : T(W_1, \Omega_{(2)}) \rightarrow T(W_1, \Omega_{(2)})$$

$$u \mapsto \sum_{i \in \mathbb{Z}} a(i) u x^{-i-1}. \quad (5.1)$$

We note that for $a \in V$

$$Y_{T(W_1, \Omega_{(2)})}(a, x)(T(V[t, t^{-1}]) \otimes_{\mathbb{C}} \Omega_{(2)}) \subset (T(V[t, t^{-1}]) \otimes_{\mathbb{C}} \Omega_{(2)})[[x, x^{-1}]],$$

$$Y_{T(W_1, \Omega_{(2)})}(a, x)(F(W_1, \Omega_{(2)})) \subset F(W_1, \Omega_{(2)})[[x, x^{-1}]] \quad (5.2)$$

and for $u \in W_1$

$$Y_{T(W_1, \Omega_{(2)})}(u, x)(T(V[t, t^{-1}]) \otimes_{\mathbb{C}} \Omega_{(2)}) \subset F(W_1, \Omega_{(2)})[[x, x^{-1}]]. \quad (5.3)$$

For homogeneous $a^1, \dots, a^{n-2} \in V$, homogeneous $u \in W_1$, $v \in \Omega_{(2)}$, $m_1, \dots, m_{n-2}, i \in \mathbb{Z}$, and $s \in \{1, \dots, n-2\}$, we define the *degree* of

$$H = a^1(m_1) \cdots a^s(m_s) u(i) a^{s+1}(m_{s+1}) \cdots a^{n-2}(m_{n-2}) v \in F(W_1, \Omega_{(2)}) \quad (5.4)$$

by

$$\deg H = \sum_{j=1}^{n-2} (\text{wt } a_j - m_j - 1) + (\deg u - i - 1). \quad (5.5)$$

For $n \in \mathbb{Z}$, we denote by $F(W_1, \Omega_{(2)})(n)$ the set of all elements in $F(W_1, \Omega_{(2)})$ with degree n . Then, we have

$$F(W_1, \Omega_{(2)}) = \bigoplus_{n \in \mathbb{Z}} F(W_1, \Omega_{(2)})(n). \quad (5.6)$$

Definition 5.1. Let $J(W_1, \Omega_{(2)})$ be the subspace of $F(W_1, \Omega_{(2)})$ generated by the following elements:

- (1) all elements in $\oplus_{n < 0} F(W_1, \Omega_{(2)})(n)$.
 (2) For homogeneous $a, b \in V$, $p \in T(V[t, t^{-1}])$, homogeneous $q \in F(W_1, \Omega_{(2)})$, and $l, m, n \in \mathbb{Z}$,

$$\begin{aligned}
 & p \left(\sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i}b)(m+n-i) \right. \\
 & - \sum_{i=0}^{\text{wt } b - n - 1 + \deg q} (-1)^i \binom{l}{i} a(m+l-i)b(n+i) \\
 & \left. - (-1)^{l+1} \sum_{i=0}^{\text{wt } a - m - 1 + \deg q} (-1)^i \binom{l}{i} b(n+l-i)a(m+i) \right) q.
 \end{aligned} \tag{5.7}$$

- (3) For $p \in T(V[t, t^{-1}])$, $q \in F(W_1, \Omega_{(2)})$, and $n \in \mathbb{Z}$,

$$p(\mathbf{1}(n) - \delta_{n,-1})q. \tag{5.8}$$

- (4) For homogeneous $a \in V$, $v \in \Omega_{(2)}$, and $p \in T(V[t, t^{-1}]) \otimes_{\mathbb{C}} W_1[t, t^{-1}] \otimes_{\mathbb{C}} T(V[t, t^{-1}])$,

$$p(a(\text{wt } a - 1) - o(a))v. \tag{5.9}$$

- (5) For homogeneous $a, b \in V$, homogeneous $q \in T(V[t, t^{-1}]) \otimes \Omega_{(2)}$, $p \in T(V[t, t^{-1}]) \otimes_{\mathbb{C}} W_1[t, t^{-1}] \otimes_{\mathbb{C}} T(V[t, t^{-1}])$, and $l, m, n \in \mathbb{Z}$,

$$\begin{aligned}
 & p \left(\sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i}b)(m+n-i) \right. \\
 & - \sum_{i=0}^{\text{wt } b - n - 1 + \deg q} (-1)^i \binom{l}{i} a(m+l-i)b(n+i) \\
 & \left. - (-1)^{l+1} \sum_{i=0}^{\text{wt } a - m - 1 + \deg q} (-1)^i \binom{l}{i} b(n+l-i)a(m+i) \right) q.
 \end{aligned} \tag{5.10}$$

- (6) For $p \in T(V[t, t^{-1}]) \otimes_{\mathbb{C}} W_1[t, t^{-1}] \otimes_{\mathbb{C}} T(V[t, t^{-1}])$, $q \in T(V[t, t^{-1}]) \otimes \Omega_{(2)}$, and $n \in \mathbb{Z}$,

$$p(\mathbf{1}(n) - \delta_{n,-1})q. \tag{5.11}$$

- (7) For homogeneous $a \in V$, homogeneous $u \in W_1$, homogeneous $q \in T(V[t, t^{-1}]) \otimes \Omega_{(2)}$, $p \in T(V[t, t^{-1}])$, and $l, m, n \in \mathbb{Z}$,

$$\begin{aligned}
& p \left(\sum_{i=0}^{\infty} \binom{m}{i} (a_{l+i}u)(m+n-i) \right. \\
& - \sum_{i=0}^{\deg u - n - 1 + \deg q} (-1)^i \binom{l}{i} a(m+l-i)u(n+i) \\
& \left. - (-1)^{l+1} \sum_{i=0}^{\text{wt } a - m - 1 + \deg q} (-1)^i \binom{l}{i} u(n+l-i)a(m+i) \right) q.
\end{aligned} \tag{5.12}$$

Since $J(W_1, \Omega_{(2)})$ is generated by homogeneous elements, we have

$$J(W_1, \Omega_{(2)}) = \bigoplus_{n \in \mathbb{Z}} (J(W_1, \Omega_{(2)}) \cap F(W_1, \Omega_{(2)})(n)). \tag{5.13}$$

We set

$$S(W_1, \Omega_{(2)}) = F(W_1, \Omega_{(2)}) / J(W_1, \Omega_{(2)}) \tag{5.14}$$

and

$$S(W_1, \Omega_{(2)})(n) = F(W_1, \Omega_{(2)})(n) + J(W_1, \Omega_{(2)}) \tag{5.15}$$

for $n \in \mathbb{Z}$. We have $S(W_1, \Omega_{(2)})(n) = 0$ for $n < 0$ by Definition 5.1 (1) and

$$S(W_1, \Omega_{(2)}) = \bigoplus_{n=0}^{\infty} S(W_1, \Omega_{(2)})(n). \tag{5.16}$$

We shall use elements of $T(W_1, \Omega_{(2)})$ to represent elements of $S(W_1, \Omega_{(2)})$. For $a \in V$, $Y_{S(W_1, \Omega_{(2)})}(a, x)$ denotes the map $S(W_1, \Omega_{(2)}) \rightarrow S(W_1, \Omega_{(2)})(\langle x \rangle)$ induced by $Y_{T(W_1, \Omega_{(2)})}(\cdot, x)$, namely

$$\begin{aligned}
Y_{S(W_1, \Omega_{(2)})}(a, x) : S(W_1, \Omega_{(2)}) & \rightarrow S(W_1, \Omega_{(2)})(\langle x \rangle) \\
u & \mapsto \sum_{i \in \mathbb{Z}} a(i) u x^{-i-1}.
\end{aligned} \tag{5.17}$$

By definition, $S(W_1, \Omega_{(2)})$ is an \mathbb{N} -graded weak V -module.

For a vector space U , we define

$$U_{\{y_1, \dots, y_n\}} = U[[y_i - y_j \mid 1 \leq i < j \leq n]][(y_i - y_j)^{-1} \mid 1 \leq i < j \leq n]. \quad (5.18)$$

For distinct $i, j \in \{1, \dots, n\}$, let

$$\iota_{(i,j)} : U_{\{y_1, \dots, y_n\}} \rightarrow U_{\{y_1, \dots, \widehat{y_i}, \dots, y_n\}}((y_i - y_j)) \quad (5.19)$$

be a linear map, where $\widehat{y_i}$ denotes the omission of the term y_i , defined by $\iota_{(i,j)}(u) = u$ for $u \in U$ and

$$\iota_{(i,j)}(y_k - y_l)^m = \begin{cases} (y_k - y_l)^m, & \text{if } k, l \neq i, \\ \sum_{s=0}^{\infty} \binom{m}{s} (y_j - y_l)^{m-s} (y_i - y_j)^s, & \text{if } k = i, \text{ and} \\ \sum_{s=0}^{\infty} \binom{m}{s} (y_k - y_j)^{m-s} (-y_i + y_j)^s, & \text{if } l = i \end{cases} \quad (5.20)$$

for distinct $k, l \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$. For $i_1, j_1, i_2, j_2 \in \{1, \dots, n\}$ such that $i_1 \neq j_1$, $i_2 \neq j_2$, and $j_1 \neq i_2$, we define a map

$$\iota_{(i_1, j_1), (i_2, j_2)} : U_{\{y_1, \dots, y_n\}} \rightarrow U_{\{y_1, \dots, \widehat{y_{i_1}}, \dots, \widehat{y_{i_2}}, \dots, y_n\}}((y_{i_1} - y_{j_1}))((y_{i_2} - y_{j_2})) \quad (5.21)$$

as follows: for $f \in U_{\{y_1, \dots, y_n\}}$, writing

$$\begin{aligned} \iota_{(i_2, j_2)} f &= \sum_{k \in \mathbb{Z}} f_k (y_{i_2} - y_{j_2})^k, \quad f_k \in U_{\{y_1, \dots, \widehat{y_{i_2}}, \dots, y_n\}} \\ &\in U_{\{y_1, \dots, \widehat{y_{i_2}}, \dots, y_n\}}((y_{i_2} - y_{j_2})), \end{aligned} \quad (5.22)$$

we define

$$\begin{aligned} \iota_{(i_1, j_1), (i_2, j_2)} f &= \sum_{k \in \mathbb{Z}} (\iota_{(i_1, j_1)} f_k) (y_{i_2} - y_{j_2})^k \\ &\in U_{\{y_1, \dots, \widehat{y_{i_1}}, \dots, \widehat{y_{i_2}}, \dots, y_n\}}((y_{i_1} - y_{j_1}))((y_{i_2} - y_{j_2})). \end{aligned} \quad (5.23)$$

By the same manner we inductively define a map

$$\iota_{(i_1, j_1), \dots, (i_k, j_k)} : U_{\{y_1, \dots, y_n\}} \rightarrow U_{\{y_1, \dots, \widehat{y_{i_1}}, \dots, \widehat{y_{i_k}}, \dots, y_n\}}((y_{i_1} - y_{j_1})) \cdots ((y_{i_k} - y_{j_k})) \quad (5.24)$$

for $i_1, j_1, \dots, i_k, j_k \in \{1, \dots, n\}$ such that $i_m \neq j_m$ and $j_m \notin \{i_{m+1}, \dots, i_k\}$ for all $m = 1, \dots, k$. We note that for distinct $i_1, \dots, i_n \in \{1, \dots, n\}$

$\iota_{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)}$ and $\iota_{(i_2, j_2), \dots, (i_n, j_n)}$ are the same maps on $U_{\{y_1, \dots, y_n\}}$ by definition.

Let U be a vector space over \mathbb{C} and $h \in \mathbb{Z}$. we say $p \in U[[y_i - y_j]^{\pm 1} \mid 1 \leq i < j \leq n]]$ is *homogeneous of total degree h* if all the terms appearing in it with nonzero coefficients have the same total degree h . We note that for every distinct $i, j \in \{1, \dots, n\}$, $p \in U_{\{y_1, \dots, y_n\}}$ is homogeneous of total degree h if and only if so is $\iota_{(i, j)}p$.

We write

$$V = \bigoplus_{i=\Delta}^{\infty} V_i \quad (5.25)$$

where $V_i = \{a \in V \mid L(0)a = ia\}$. Let M be a weak V -module. For homogeneous $a^1, \dots, a^{n-1} \in V$ and homogeneous $u \in M$, standard arguments (cf. [12, Sections 3.2–3.4] and [17, Lemma 2.4]) show that there exists

$$\begin{aligned} & \hat{Y}_M(a^1, \dots, a^{n-1}, u | y_1, \dots, y_n) \\ & \in \prod_{1 \leq s < t \leq n-1} (y_s - y_t)^{-\text{wt } a^s - \text{wt } a^t + \Delta} \prod_{1 \leq m < n} (y_m - y_n)^{-\text{wt } a^m - \deg u} \\ & \quad \times M[[y_i - y_j \mid 1 \leq i < j \leq n]] \\ & \subset M_{\{y_1, \dots, y_n\}} \end{aligned} \quad (5.26)$$

such that

$$\begin{aligned} & \iota_{(1, n), (2, n), \dots, (n-1, n)} \hat{Y}_M(a^1, a^2, \dots, a^{n-1}, u | y_1, \dots, y_n) \\ & = Y_M(a^1, y_1 - y_n) Y_M(a^2, y_2 - y_n) \cdots Y_M(a^{n-1}, y_{n-1} - y_n) u \\ & \in M((y_1 - y_n)((y_2 - y_n)) \cdots ((y_{n-1} - y_n))). \end{aligned} \quad (5.27)$$

We note that

$$\hat{Y}_M(a, u | y_1, y_2) = Y_M(a, y_1 - y_2) u \quad (5.28)$$

for $a \in V$ and $u \in M$. Standard arguments (cf. [12, Sections 3.2–3.4] and [17, Lemma 2.4]) also show that

$$\begin{aligned} & \hat{Y}_M(a^{\sigma(1)}, \dots, a^{\sigma(n-1)}, u | y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, y_n) \\ & = \hat{Y}_M(a^1, \dots, a^{n-1}, u | y_1, \dots, y_n) \end{aligned} \quad (5.29)$$

for an arbitrary permutation σ of $\{1, \dots, n-1\}$,

$$\begin{aligned} & \iota_{(i, i+1)} \hat{Y}_M(a^1, \dots, a^{n-1}, u | y_1, \dots, y_n) \\ & = \hat{Y}_M(a^1, \dots, a^{i-1}, Y(a^i, y_i - y_{i+1}) a^{i+1}, a^{i+2}, \dots, a^{n-1}, u | y_1, \dots, \hat{y}_i, \dots, y_n) \\ & \in M_{\{y_1, \dots, \hat{y}_i, \dots, y_n\}}((y_i - y_{i+1})) \end{aligned} \quad (5.30)$$

for $i = 1, \dots, n-2$, and

$$\begin{aligned} & \iota_{(n-1,n)} \hat{Y}_M(a^1, \dots, a^{n-1}, u|y_1, \dots, y_n) \\ &= \hat{Y}_M(a^1, \dots, a^{n-2}, Y_M(a^{n-1}, y_{n-1} - y_n)u|y_1, \dots, y_{n-2}, y_n) \\ &\in M_{\{y_1, \dots, y_{n-2}, y_n\}}((y_{n-1} - y_n)). \end{aligned} \quad (5.31)$$

For $n \in \mathbb{N}$, $V^{\times n}$ denotes the n times direct product of V . Let

$$f_n : V^{\times n-2} \times W_1 \times \Omega_{(2)} \rightarrow U_{\{y_1, \dots, y_n\}}, \quad n = 2, 3, \dots \quad (5.32)$$

be a sequence of maps which satisfies the following conditions: let $a^1, \dots, a^{n-2} \in V$, $u \in W_1$, and $v \in \Omega_{(2)}$.

(1) For an arbitrary permutation σ of $\{1, \dots, n-2\}$,

$$\begin{aligned} & f_n(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_{n-2}, y_{n-1}, y_n) \\ &= f_n(a^{\sigma(1)}, \dots, a^{\sigma(n-2)}, u, v|y_{\sigma(1)}, \dots, y_{\sigma(n-2)}, y_{n-1}, y_n). \end{aligned} \quad (5.33)$$

(2) For $i = 1, \dots, n-3$,

$$\begin{aligned} & \iota_{(i,i+1)} f_n(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_n) \\ &= f_{n-1}(a^1, \dots, Y(a^i, y_i - y_{i+1})a^{i+1}, \dots, a^{n-2}, u, v|y_1, \dots, \hat{y}_i, \dots, y_n) \\ &\in U_{\{y_1, \dots, \hat{y}_i, \dots, y_n\}}((y_i - y_{i+1})). \end{aligned} \quad (5.34)$$

(3)

$$\begin{aligned} & \iota_{(n-2,n-1)} f_n(a^1, \dots, a^{n-3}, a^{n-2}, u, v|y_1, \dots, y_n) \\ &= f_{n-1}(a^1, \dots, a^{n-3}, Y_{W_1}(a^{n-2}, y_{n-2} - y_{n-1})u, v|y_1, \dots, y_{n-3}, y_{n-1}, y_n) \\ &\in U_{\{y_1, \dots, y_{n-3}, y_{n-1}, y_n\}}((y_{n-2} - y_{n-1})). \end{aligned} \quad (5.35)$$

(4) If a^{n-2} is homogeneous, then the coefficient of $(y_{n-2} - y_n)^{-\text{wt } a^{n-2}}$ in

$$\iota_{(n-2,n)} f_n(a^1, \dots, a^{n-3}, a^{n-2}, u, v|y_1, \dots, y_n)$$

is equal to

$$f_{n-1}(a^1, \dots, a^{n-3}, u, o(a^{n-2})v|y_1, \dots, y_{n-3}, y_{n-1}, y_n). \quad (5.36)$$

We define a map $\Phi : F(W_1, \Omega_{(2)}) \rightarrow U$ by

$$\begin{aligned}
& \Phi(Y_{T(W_1, \Omega_{(2)})}(a^1, y_1 - y_n) \cdots Y_{T(W_1, \Omega_{(2)})}(a^s, y_s - y_n) \\
& \quad \times Y_{T(W_1, \Omega_{(2)})}(u, y_{n-1} - y_n) \\
& \quad \times Y_{T(W_1, \Omega_{(2)})}(a^{s+1}, y_{s+1} - y_n) \cdots Y_{T(W_1, \Omega_{(2)})}(a^{n-2}, y_{n-2} - y_n)v) \\
& = \iota_{(1,n), \dots, (s,n), (n-1,n), (s+1,n), \dots, (n-2,n)} f_n(a^1, \dots, a^{n-2}, u, v | y_1, \dots, y_n)
\end{aligned} \tag{5.37}$$

for $a^1, \dots, a^{n-2} \in V, u \in W_1, v \in \Omega_{(2)}$, and $s = 1, \dots, n-2$.

Lemma 5.2. *With the notation above, $\Phi(J(W_1, \Omega_{(2)})) = 0$ and therefore the map $\Phi : T(W_1, \Omega_{(2)}) \rightarrow U$ induces a map $S(W_1, \Omega_{(2)}) \rightarrow U$ which denoted by the same symbol:*

$$\begin{aligned}
\Phi : \quad S(W_1, \Omega_{(2)}) & \rightarrow U \\
u & \mapsto \Phi(u).
\end{aligned} \tag{5.38}$$

Proof. We simply write $Y = Y_{T(W_1, \Omega_{(2)})}$. We only show that the images of elements of the forms (5.7) and (5.8) in Definition 5.1 vanish. We can show the images of the other elements in Definition 5.1 vanish in the same manner.

For $i, s = 1, \dots, n-3$ with $i+1 < s$, defining

$$\begin{aligned}
P &= Y(a^1, y_1 - y_n) \cdots Y(a^{i-1}, y_{i-1} - y_n), \\
Q &= Y(a^{i+2}, y_{i+2} - y_n) \cdots Y(a^s, y_s - y_n) Y(u, y_{n-1} - y_n) \\
& \quad \times Y(a^{s+1}, y_{s+1} - y_n) \cdots Y(a^{n-2}, y_{n-2} - y_n)v, \\
\iota_1 &= \iota_{(1,n), \dots, (i-1,n)}, \text{ and} \\
\iota_2 &= \iota_{(i+2,n), \dots, (s,n), (n-1,n), (s+1,n), \dots, (n-2,n)},
\end{aligned} \tag{5.39}$$

we have

$$\begin{aligned}
& \iota_1 \circ \iota_{(i,n), (i+1,n)} \circ \iota_2(f_n(a^1, \dots, a^{n-2}, u, v | y_1, \dots, y_n)) \\
& = f_2(PY(a^i, y_i - y_n)Y(a^{i+1}, y_{i+1} - y_n)Q), \\
& \iota_1 \circ \iota_{(i+1,n), (i,n)} \circ \iota_2(f_n(a^1, \dots, a^{n-2}, u, v | y_1, \dots, y_n)) \\
& = f_2(PY(a^{i+1}, y_{i+1} - y_n)Y(a^i, y_i - y_n)Q), \quad \text{and} \\
& \iota_1 \circ \iota_{(i+1,n), (i, i+1)} \circ \iota_2(f_n(a^1, \dots, a^{n-2}, u, v | y_1, \dots, y_n)) \\
& = f_2(PY(Y(a^i, y_i - y_{i+1})a^{i+1}, y_{i+1} - y_n)Q).
\end{aligned} \tag{5.40}$$

Let p be an arbitrary coefficient of P and q an arbitrary coefficient of Q . Taking $a^i = a$ and $a^{i+1} = b$ in (5.40), we have (5.7) by standard arguments (cf. [12, Sections 3.2–3.4] and [17, Lemma 2.4]).

For $i = 1, \dots, n-3$, we have

$$\begin{aligned} & \iota_{(i,i+1)}(f_n(\dots, a^{i-1}, \mathbf{1}, a^{i+1}, \dots | y_1, \dots, y_n)) \\ &= f_{n-1}(\dots, a^{i-1}, Y(Y(\mathbf{1}, y_i - y_{i+1})a^{i+1}, \dots | y_1, \dots, \widehat{y}_i, \dots, y_n)) \\ &= f_{n-1}(\dots, a^{i-1}, a^{i+1}, \dots | y_1, \dots, \widehat{y}_i, \dots, y_n) \\ &\in T(W_1, \Omega_{(2)})_{\{y_1, \dots, \widehat{y}_i, \dots, y_n\}}. \end{aligned} \quad (5.41)$$

Since $T(W_1, \Omega_{(2)})_{\{y_1, \dots, \widehat{y}_i, \dots, y_n\}} \subset T(W_1, \Omega_{(2)})_{\{y_1, \dots, y_n\}}$ and $\iota_{(i,i+1)}$ is injective, we have

$$\begin{aligned} & f_n(\dots, a^{i-1}, \mathbf{1}, a^{i+1}, \dots | y_1, \dots, y_n) \\ &= f_{n-1}(\dots, a^{i-1}, a^{i+1}, \dots | y_1, \dots, \widehat{y}_i, \dots, y_n) \end{aligned} \quad (5.42)$$

and therefore (5.8) by (5.37). \square

For $\Phi(\cdot, x) \in I_{\mathbb{Z}}\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right)$ and homogeneous $u \in W_1$, we denote $\Phi(u; \deg u - 1)$ by $o^\Phi(u)$ and extend $o^\Phi(u)$ for an arbitrary $u \in W_1$ by linearity. The map

$$\begin{aligned} o^\Phi : W_1 \otimes W_2(0) &\rightarrow W_3(0) \\ u \otimes v &\mapsto o^\Phi(u)v \end{aligned} \quad (5.43)$$

induces an $A(V)$ -module homomorphism $A(W_1) \otimes_{A(V)} W_2(0) \rightarrow W_3(0)$ which denoted by the same symbol:

$$\begin{aligned} o^\Phi : A(W_1) \otimes_{A(V)} W_2(0) &\rightarrow W_3(0) \\ u \otimes v &\mapsto o^\Phi(u)v. \end{aligned} \quad (5.44)$$

We get a map

$$\begin{aligned} I_{\mathbb{Z}}\left(\begin{smallmatrix} W_3 \\ W_1 \ W_2 \end{smallmatrix}\right) &\rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)) \\ \Phi(\cdot, x) &\mapsto o^\Phi. \end{aligned} \quad (5.45)$$

The following is the main result.

Theorem 5.3. *For an \mathbb{N} -graded weak V -module W_1 and two left $A(V)$ -modules $\Omega_{(2)}$ and $\Omega_{(3)}$, the map*

$$I_{\mathbb{Z}} \left(\begin{array}{c} S(\Omega_{(3)}^*)' \\ W_1 \ S(\Omega_{(2)}) \end{array} \right) \rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} \Omega_{(2)}, \Omega_{(3)})$$

$$\Phi(\cdot, x) \mapsto o^\Phi \quad (5.46)$$

is a linear isomorphism.

Proof. The same argument as in the proof of [13, Proposition 2.10] shows the map (5.46) is injective. Let $\varphi : A(W_1) \otimes_{A(V)} \Omega_{(2)} \rightarrow \Omega_{(3)}$ be an $A(V)$ -module homomorphism. For homogeneous $u \in W_1$ and $v \in \Omega_{(2)}$, we define

$$\varphi(u, v|y_1, y_2) = \varphi(u \otimes v)(y_1 - y_2)^{-\deg u} \quad (5.47)$$

and extend $\varphi(u, v|y_1, y_2)$ for an arbitrary $u \in W_1$ by linearity. For $a^1, \dots, a^{n-2} \in V$, $u \in W_1$, $v \in \Omega_{(2)}$, and $i, j \in \{1, \dots, n\}$ with $i < j$, as temporary notation let us put

$$\Gamma_{ij} = \begin{cases} \text{wt } a^i + \text{wt } a^j - \Delta, & \text{if } 1 \leq i < j \leq n-2, \\ \text{wt } a^i + \deg u, & \text{if } 1 \leq i \leq n-2 \text{ and } j = n-1, \\ \text{wt } a^i, & \text{if } 1 \leq i \leq n-2 \text{ and } j = n, \\ \deg u, & \text{if } i = n-1 \text{ and } j = n \end{cases} \quad (5.48)$$

and

$$g = \prod_{1 \leq i < j \leq n} (y_i - y_j)^{\Gamma_{ij}} \varphi(\hat{Y}_{W_1}(a^1, \dots, a^{n-2}, u|y_1, \dots, y_{n-2}, y_{n-1}), v|y_{n-1}, y_n) \quad (5.49)$$

where Δ is defined in (5.25). Since

$$\prod_{1 \leq i < j \leq n-1} (y_i - y_j)^{\Gamma_{ij}} \hat{Y}_{W_1}(a^1, \dots, a^{n-2}, u|y_1, \dots, y_{n-2}, y_{n-1})$$

$$\in W_1[[y_i - y_j \mid 1 \leq i < j \leq n-1]] \quad (5.50)$$

by (5.26), we have

$$g \in (\Omega_{(3)}((y_{n-1} - y_n)))[[y_i - y_j \mid 1 \leq i < j \leq n-1]]$$

$$\times [(y_1 - y_n)^{\pm 1}, \dots, (y_{n-2} - y_n)^{\pm 1}]. \quad (5.51)$$

We note that g is homogeneous of total degree

$$\Gamma = \sum_{1 \leq i < j \leq n} \Gamma_{ij} - \sum_{i=1}^{n-2} \text{wt } a^i - \deg u. \quad (5.52)$$

For an arbitrary permutation σ of $\{1, \dots, n-2\}$, by (5.50) and

$$\begin{aligned} & \iota_{(1,n-1), \dots, (n-2,n-1)} \prod_{i=1}^{n-1} (y_i - y_n)^{\Gamma_{in}} \\ &= \iota_{(\sigma(1),n-1), \dots, (\sigma(n-2),n-1)} \prod_{i=1}^{n-1} (y_i - y_n)^{\Gamma_{in}}, \end{aligned} \quad (5.53)$$

we have

$$\iota_{(1,n-1), \dots, (n-2,n-1)} g = \iota_{(\sigma(1),n-1), \dots, (\sigma(n-2),n-1)} g \quad (5.54)$$

and therefore

$$\begin{aligned} & \iota_{(1,n-1), \dots, (n-2,n-1)} \left(\prod_{1 \leq i < j \leq n} (y_i - y_j)^{\Gamma_{ij}} \right. \\ & \quad \times \varphi(Y_{W_1}(a^1, y_1 - y_{n-1}) \cdots Y_{W_1}(a^{n-2}, y_{n-2} - y_{n-1})u, v | y_{n-1}, y_n) \Big) \\ &= \iota_{(\sigma(1),n-1), \dots, (\sigma(n-2),n-1)} \left(\prod_{1 \leq i < j \leq n} (y_i - y_j)^{\Gamma_{ij}} \right. \\ & \quad \times \varphi(Y_{W_1}(a^{\sigma(1)}, y_{\sigma(1)} - y_{n-1}) \cdots Y_{W_1}(a^{\sigma(n-2)}, y_{\sigma(n-2)} - y_{n-1})u, v | y_{n-1}, y_n) \Big) \end{aligned} \quad (5.55)$$

by (5.27) and (5.29).

Let i_1, \dots, i_{n-2}, i_n be a sequence of integers such that

$$\begin{aligned} & 0 \neq \text{Res}_{y_{n-1}-y_n} \text{Res}_{y_1-y_{n-1}} \cdots \text{Res}_{y_{n-2}-y_{n-1}} \\ & \quad \times (y_{n-1} - y_n)^{i_n} (y_1 - y_{n-1})^{i_1} \cdots (y_{n-2} - y_{n-1})^{i_{n-1}} \\ & \quad \times \iota_{(1,n-1), \dots, (n-2,n-1)} g. \end{aligned} \quad (5.56)$$

By (5.52), we have

$$i_1 + \cdots + i_{n-2} + i_n = -\Gamma - n + 1. \quad (5.57)$$

By (5.55), we may assume that i_1 is the smallest element in $\{i_1, \dots, i_{n-2}\}$. Since

$$\begin{aligned}
& \iota_{(1,n-1), \dots, (n-2,n-1)} \prod_{1 \leq i < j \leq n} (y_i - y_j)^{\Gamma_{ij}} \\
&= \sum_{k_{12}, k_{13}, \dots, k_{n-3, n-2}=0}^{\infty} \sum_{k_{1n}, \dots, k_{n-2, n}=0}^{\infty} \left(\prod_{1 \leq i < j \leq n-2} \binom{\Gamma_{ij}}{k_{ij}} \right) \left(\prod_{i=1}^{n-2} \binom{\Gamma_{in}}{k_{in}} \right) \\
&\quad \times (y_{n-1} - y_n)^{\sum_{m=1}^{n-1} \Gamma_{mn} - \sum_{m=1}^{n-2} k_{mn}} \\
&\quad \times \prod_{j=1}^{n-2} (-1)^{\sum_{m=1}^{j-1} k_{mj}} (y_j - y_{n-1})^{k_{jn} + \sum_{m=1}^{j-1} k_{mj} + \sum_{m=j+1}^{n-1} \Gamma_{jm} - \sum_{m=j+1}^{n-2} k_{jm}} \\
&= \sum_{k_{12}, k_{13}, \dots, k_{n-3, n-2}=0}^{\infty} \sum_{k_{2n}, \dots, k_{n-2, n}=0}^{\infty} \left(\prod_{1 \leq i < j \leq n-2} \binom{\Gamma_{ij}}{k_{ij}} \right) \left(\prod_{i=2}^{n-2} \binom{\Gamma_{in}}{k_{in}} \right) \\
&\quad \times \sum_{k_{1n}=0}^{\infty} \binom{\Gamma_{1n}}{k_{1n}} (y_{n-1} - y_n)^{\Gamma_{1n} - k_{1n} + \sum_{m=2}^{n-1} \Gamma_{mn} - \sum_{m=2}^{n-2} k_{mn}} \\
&\quad \times (y_1 - y_{n-1})^{k_{1n} + \sum_{m=2}^{n-1} \Gamma_{1m} - \sum_{m=2}^{n-2} k_{1m}} \\
&\quad \times \prod_{j=2}^{n-2} (-1)^{\sum_{m=1}^{j-1} k_{mj}} (y_j - y_{n-1})^{k_{jn} + \sum_{m=1}^{j-1} k_{mj} + \sum_{m=j+1}^{n-1} \Gamma_{jm} - \sum_{m=j+1}^{n-2} k_{jm}}, \\
&\hspace{15em} (5.58)
\end{aligned}$$

the right-hand side of (5.56) can be written as a linear combination of the following elements:

$$\begin{aligned}
& \text{Res}_{y_{n-1}-y_n} \text{Res}_{y_1-y_{n-1}} \sum_{k_{1n}=0}^{\infty} \binom{\Gamma_{1n}}{k_{1n}} (y_{n-1} - y_n)^{\Gamma_{1n} - k_{1n} + d} \\
&\quad \times (y_1 - y_{n-1})^{i_1 + \sum_{m=2}^{n-1} \Gamma_{1m} - l + k_{1n}} \varphi(Y(a^1, y_1 - y_{n-1})w, v | y_{n-1}, y_n) \\
&\hspace{15em} (5.59)
\end{aligned}$$

where $l \in \mathbb{N}$, $d \in \mathbb{Z}$, and w is a homogeneous element of W_1 . We see

that (5.59) becomes

$$\begin{aligned}
& \text{Res}_{y_{n-1}-y_n} \sum_{k_{1n}=0}^{\infty} \binom{\Gamma_{1n}}{k_{1n}} \varphi(a_{i_1+\sum_{m=2}^{n-1} \Gamma_{1m}-l+k_{1n}}^1 w, v | y_{n-1}, y_n) \\
& \quad \times (y_{n-1} - y_n)^{\Gamma_{1n}-k_{1n}+d} \\
& = \text{Res}_{y_{n-1}-y_n} \sum_{k_{1n}=0}^{\infty} \binom{\Gamma_{1n}}{k_{1n}} \varphi(a_{i_1+\sum_{m=2}^{n-1} \Gamma_{1m}-l+k_{1n}}^1 w \otimes v) \\
& \quad \times (y_{n-1} - y_n)^{-\text{wt } a^1 - \deg w + i_1 + \sum_{m=2}^{n-1} \Gamma_{1m} - l + 1 + \Gamma_{1n} + d} \\
& = \text{Res}_{y_{n-1}-y_n} \varphi(\text{Res}_x(1+x)^{\Gamma_{1n}} x^{i_1+\sum_{m=2}^{n-1} \Gamma_{1m}-l} Y_{W_1}(a^1, x) w \otimes v) \\
& \quad \times (y_{n-1} - y_n)^{-\text{wt } a^1 - \deg w + i_1 + \sum_{m=2}^{n-1} \Gamma_{1m} - k + 1 + \Gamma_{1n} + d}. \tag{5.60}
\end{aligned}$$

By (2.8), (5.48), (5.56), and (5.60), we have

$$i_1 \geq -\sum_{m=2}^{n-1} \Gamma_{1m} - 1. \tag{5.61}$$

Since i_1 is the smallest element of $\{i_1, \dots, i_{n-2}\}$, we have

$$i_1 + \dots + i_{n-2} + i_n \geq -(n-2) \left(\sum_{m=2}^{n-1} \Gamma_{1m} + 1 \right) + i_n \tag{5.62}$$

and therefore

$$i_n \leq -\Gamma - n + 1 + (n-2) \left(\sum_{m=2}^{n-1} \Gamma_{1m} + 1 \right) \tag{5.63}$$

by (5.57). Thus by (5.51), we have

$$g \in \Omega_{(3)}[[y_i - y_j \mid 1 \leq i < j \leq n]][(y_1 - y_n)^{-1}, \dots, (y_{n-1} - y_n)^{-1}]. \tag{5.64}$$

Since g is homogeneous of total degree Γ , we have

$$g \in \Omega_{(3)}[(y_1 - y_n)^{\pm 1}, \dots, (y_{n-1} - y_n)^{\pm 1}] \tag{5.65}$$

and therefore there exists

$$\varphi(a^1, \dots, a^{n-2}, u, v | y_1, \dots, y_n) \in \Omega_{(3)}[(y_i - y_j)^{\pm 1} \mid 1 \leq i < j \leq n] \tag{5.66}$$

such that

$$\begin{aligned} & \iota_{(n-1,n)}\varphi(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_n) \\ &= \varphi(\hat{Y}_{W_1}(a^1, \dots, a^{n-2}, u|y_1, \dots, y_{n-2}, y_{n-1}), v|y_{n-1}, y_n) \end{aligned} \quad (5.67)$$

by the definition (5.49) of g . If one put

$$\begin{aligned} & f_n(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_n) \\ &= \varphi(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_n), \quad n = 2, 3, \dots \end{aligned} \quad (5.68)$$

for $a^1, \dots, a^{n-2} \in V, u \in W_1$ and $v \in \Omega_{(2)}$, then f_n ($n = 2, 3, \dots$) satisfy (5.33)–(5.36) by (5.67). It follows from Lemma 5.2 that there exists a map $\Phi : S(W_1, \Omega_{(2)}) \rightarrow \Omega_{(3)}$ such that

$$\begin{aligned} & \Phi(Y_{S(W_1, \Omega_{(2)})}(a^1, y_1 - y_n) \cdots, Y_{S(W_1, \Omega_{(2)})}(a^s, y_s - y_n) \\ & \quad \times Y_{S(W_1, \Omega_{(2)})}(u, y_{n-1} - y_n) \\ & \quad \times Y_{S(W_1, \Omega_{(2)})}(a^{s+1}, y_{s+1} - y_n) \cdots Y_{S(W_1, \Omega_{(2)})}(a^{n-2}, y_{n-2} - y_n)v) \\ &= \iota_{(1,n), \dots, (s,n), (n-1,n), (s+1,n), \dots, (n-2,n)}\varphi(a^1, \dots, a^{n-2}, u, v|y_1, \dots, y_n) \end{aligned} \quad (5.69)$$

for $a^1, \dots, a^{n-2} \in V, u \in W_1, v \in \Omega_{(2)}$, and $s = 1, \dots, n-2$.

We define an $A(V)$ -module homomorphism $\mu' : \Omega_{(3)}^* \rightarrow S(W_1, \Omega_{(2)})'(0)$ by

$$\langle \mu'(w'_{(3)}), u \rangle = \langle w'_{(3)}, \Phi(u) \rangle \quad (5.70)$$

for $w'_{(3)} \in \Omega_3^*$ and $u \in S(W_1, \Omega_{(2)})(0)$. By the universality of the generalized Verma module $S(\Omega_{(3)}^*)$, we have a V -module homomorphism

$$\mu' : S(\Omega_{(3)}^*) \rightarrow S(W_1, \Omega_{(2)})' \quad (5.71)$$

and therefore its dual

$$\mu'' : S(W_1, \Omega_{(2)})'' \rightarrow S(\Omega_{(3)}^*)'. \quad (5.72)$$

Restricting μ'' to $S(W_1, \Omega_{(2)})$, we have

$$\mu : S(W_1, \Omega_{(2)}) \rightarrow S(\Omega_{(3)}^*)'. \quad (5.73)$$

Then the map

$$\begin{aligned} & W_1 \otimes_{\mathbb{C}} S(\Omega_2) \rightarrow S(\Omega_{(3)}^*)'((x)) \\ & u \otimes v \mapsto \sum_{i \in \mathbb{Z}} \mu(u(i)v)x^{-i-1}. \end{aligned} \quad (5.74)$$

is the desired \mathbb{Z} -graded intertwining operator by the definition of μ . \square

The following result is a direct consequence of Theorem 5.3.

Corollary 5.4. *Let $W_i = \bigoplus_{j=0}^{\infty} W_i(j)$, $i = 1, 2, 3$ be three \mathbb{N} -graded weak V -modules such that W_2 and W_3' are generalize Verma V -modules and $\dim_{\mathbb{C}} W_3(j) < \infty$ for all $j \in \mathbb{N}$. Then, the map*

$$\begin{aligned} I_{\mathbb{Z}} \begin{pmatrix} W_3 \\ W_1 \ W_2 \end{pmatrix} &\rightarrow \text{Hom}_{A(V)}(A(W_1) \otimes_{A(V)} W_2(0), W_3(0)) \\ \Phi(\cdot, x) &\mapsto o^{\Phi} \end{aligned} \quad (5.75)$$

is a linear isomorphism.

Remark 5.5. For arbitrary $c, h \in \mathbb{C}$, $M_{c,h}$ denotes the Verma module for the Virasoro algebra of central charge c with lowest weight h and M_c denotes the quotient space of $M(c, 0)$ by the submodule generated by $L(-1)\mathbf{1}$ where $\mathbf{1}$ is a lowest weight vector of weight 0 for $M(c, 0)$. Then, M_c is a vertex operator algebra and $M_{c,h}$ is an M_c -module (cf. [12]). If $V = M_c$ and $W_1 = M_{c,h}$ in Theorem 5.3, then using the same computation as in [13, Section 2], we can describe the right-hand side of (5.46) as follows. It is shown in [7, Section 4] (see also [6, Proposition 3.1]) that

$$\begin{aligned} A(M_c) &\rightarrow \mathbb{C}[t] \\ (L(-2) + L(-1))^n &\mapsto t^n, \quad n \in \mathbb{N} \end{aligned} \quad (5.76)$$

and

$$\begin{aligned} A(M_{c,h}) &\rightarrow \mathbb{C}[t_1, t_2] \\ (L(-2) + 2L(-1) + L(0))^m ((L(-2) + L(-1))^n v_h) &\mapsto t_1^m t_2^n, \quad m, n \in \mathbb{N} \end{aligned} \quad (5.77)$$

are isomorphisms where v_h is a non-zero element of $M(c, h)$ with weight h and the $\mathbb{C}[t]$ -bimodule structure on $\mathbb{C}[t_1, t_2]$ is given by

$$\begin{aligned} t^n \cdot f(t_1, t_2) &= t_1^n f(t_1, t_2) \text{ and} \\ f(t_1, t_2) \cdot t^n &= t_2^n f(t_1, t_2) \end{aligned} \quad (5.78)$$

for $n \in \mathbb{N}$ and $f(t_1, t_2) \in \mathbb{C}[t_1, t_2]$. Thus, we have

$$\begin{aligned} &\text{Hom}_{A(M_c)}(A(M_{c,h}) \otimes_{A(M_c)} \Omega_{(2)}, \Omega_{(3)}) \\ &\cong \text{Hom}_{\mathbb{C}[t]}(\mathbb{C}[t_1, t_2] \otimes_{\mathbb{C}[t]} \Omega_{(2)}, \Omega_{(3)}) \\ &\cong \text{Hom}_{\mathbb{C}}(\Omega_{(2)}, \Omega_{(3)}). \end{aligned} \quad (5.79)$$

6 Notation

$M[x, x^{-1}]_{[p,q]}$	$= \{\sum_{i=p}^q u_i x^i \mid u_p, u_{p+1}, \dots, u_q \in M\}.$
$A(V)$	the Zhu algebra of a vertex operator algebra V .
$A(M)$	the $A(V)$ -bimodule associated with a weak V -module M .
$\text{Res}_x f(x)$	$= f_{-1}$ for $f(x) = \sum_{i \in \mathbb{Z}} f_i x^i$.
$I_{\binom{W_3}{W_1 W_2}}$	the space of all intertwining operators of type $\binom{W_3}{W_1 W_2}$.
$I_{\log} \binom{W_3}{W_1 W_2}$	the space of all logarithmic intertwining operators of type $\binom{W_3}{W_1 W_2}$.
$I_{\mathbb{Z}} \binom{W_3}{W_1 W_2}$	the space of all \mathbb{Z} -graded intertwining operators of type $\binom{W_3}{W_1 W_2}$.
$\Omega_{(2)}, \Omega_{(3)}$	left $A(V)$ -modules.
Δ	$V = \bigoplus_{i \in \Delta} V_i$.
$T(U)$	the tensor algebra of a vector space U .
$T(W_1, \Omega_{(2)})$	$= T((V \oplus W_1 \oplus \Omega_{(2)})[t, t^{-1}]).$
$F(W_1, \Omega_{(2)})$	$= T(V[t, t^{-1}]) \otimes_{\mathbb{C}} W_1[t, t^{-1}] \otimes_{\mathbb{C}} T(V[t, t^{-1}]) \otimes_{\mathbb{C}} \Omega_{(2)}.$
$S(U) = \bigoplus_{j=0}^{\infty} S(U)(j)$	the generalized Verma module with $S(U)(0) = U$ where U is a left $A(V)$ -module.
$J(W_1, \Omega_{(2)})$	Definition 5.1.
$S(W_1, \Omega_{(2)})$	$= F(W_1, \Omega_{(2)})/J(W_1, \Omega_{(2)}), (5.14).$
$U_{\{y_1, \dots, y_n\}}$	$= U[[y_i - y_j \mid 1 \leq i < j \leq n]][(y_i - y_j)^{-1} \mid 1 \leq i < j \leq n], (5.18).$
$\iota_{(i,j)}$	a linear map $U_{\{y_1, \dots, y_n\}} \rightarrow U_{\{y_1, \dots, \widehat{y}_i, \dots, y_n\}}((y_i - y_j))$ defined by (5.19).
$\iota_{(i_1, j_1), \dots, (i_k, j_k)}$	(5.23) and (5.24).
W^*	$= \text{Hom}_{\mathbb{C}}(W, \mathbb{C}).$
W'	$= \bigoplus_{i=0}^{\infty} \text{Hom}_{\mathbb{C}}(W(i), \mathbb{C})$ for $W = \bigoplus_{i=0}^{\infty} W(i).$

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References

- [1] T. Abe, Fusion rules for the free bosonic orbifold vertex operator algebra, *J. Algebra* **229** (2000), 333–374.
- [2] T. Abe, Fusion rules for the charge conjugation orbifold, *J. Algebra* **242** (2001), 624–655.
- [3] T. Abe, C. Dong and H. Li, Fusion rules for the vertex operator algebras $M(1)^+$ and V_L^+ . *Comm. Math. Phys.* **253** (2005), 171–219.

- [4] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Nat. Acad. Sci. U.S.A.* **83** (1986), 3068–3071.
- [5] C. Dong, H.S. Li and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.
- [6] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Algebraic groups and their generalizations: quantum and infinite-dimensional methods* (University Park, PA, 1991), 295–316, Proc. Sympos. Pure Math., **56**, Part 2, Amer. Math. Soc., Providence, RI, 1994.
- [7] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.* **66** (1992), 123–168.
- [8] I. B. Frenkel, Y. Huang and J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Mem. Amer. Math. Soc. **104**, 1993.
- [9] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, 1988.
- [10] Y. Z. Huang, J. Lepowsky and L. Zhang, A logarithmic generalization of tensor product theory for modules for a vertex operator algebra, *Internat. J. Math.* **17** (2006), 975–1012.
- [11] Y. Z. Huang and J. Yang, Logarithmic intertwining operators and associative algebras, *J. Pure Appl. Algebra* **216** (2012), 1467–1492.
- [12] J. Lepowsky and H. S. Li, *Introduction to Vertex Operator Algebras and their Representations*, Progress in Mathematics, **227**, Birkhauser Boston, Inc., Boston, MA, 2004.
- [13] H. S. Li, Determining fusion rules by $A(V)$ -modules and bimodules, *J. Algebra* **212** (1999), 515–556.
- [14] A. Milas, Weak modules and logarithmic intertwining operators for vertex operator algebras, *Recent developments in infinite-dimensional Lie algebras and conformal field theory* (Charlottesville, VA, 2000), 201–225, Contemp. Math., **297**, Amer. Math. Soc., Providence, RI, 2002.
- [15] A. Milas, Logarithmic intertwining operators and vertex operators, *Comm. Math. Phys.* **277** (2008), 497–529.

- [16] K. Tanabe, On intertwining operators and finite automorphism groups of vertex operator algebras, *J. Algebra* **287** (2005), 174–198.
- [17] K. Tanabe, A generalization of twisted modules over vertex algebras, *Journal of the Mathematical Society of Japan* **67** (2015), 1109–1146.
- [18] K. Tanabe and H. Yamada, The fixed point subalgebra of a lattice vertex operator algebra by an automorphism of order three, *Pacific Journal of Mathematics* **230** (2007), 469–510.
- [19] K. Tanabe and H. Yamada, Fixed point subalgebras of lattice vertex operator algebras by an automorphism of order three, *Journal of the Mathematical Society of Japan* **65** (2013), 1169–1242.
- [20] A. Tsuchiya and Y. Kanie, Vertex operators in conformal field theory on \mathbf{P}^1 and monodromy representations of braid group, *Conformal Field Theory and Solvable Lattice Models*, Advanced Studies in Pure Math., **16**, Kinokuniya, Tokyo, (1988), 297–372.
- [21] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.